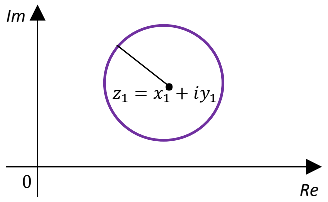
Complex Numbers Cheat Sheet

Loci in the complex plane

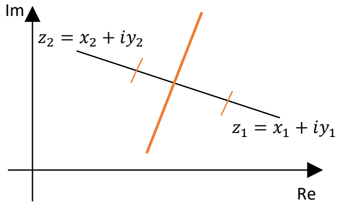
As seen previously, an Argand diagram is a graph where the x-axis represents the real numbers and the y-axis represents the imaginary numbers. We can use complex numbers to describe a locus of points on an Argand diagram.

Given a complex number $z_1 = x_1 + iy_1$:



- The locus of points z on an Argand diagram such that $|z - z_1| = r$, is a circle centred at (x_1, y_1) with radius r. You should already know that the Cartesian equation of a circle is $(x - x_1)^2 + (y - y_1)^2$
- The locus of points z on an Argand diagram such that $\arg(z - z_1) = \theta$. is a half-line from (but not including) the fixed point z_1. The open circle should be plotted at z_1 on an Argand diagram, making an angle θ with the real axis.

Given two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$:



- The locus of points z on an Argand diagram such that $|z - z_1| = |z - z_2|$ is the perpendicular bisector of the line segment joining z_1 and z_2.

You also need to be able to find the locus of a set of points whose distances from two fixed points are in a constant ratio. Although not intuitive, the locus of these points is a circle most of the time, which can be shown by forming an equation. For example, the locus points that are twice the distance from 1 as they are from $4 + 5i$ can be written as $|z - 1| = 2|z - (4 + 5i)|$. This can be rearranged into Cartesian form to find the centre and radius of the circle.

- The locus of points z that satisfy $|z - a| = k|z - b|$, where $a, b \in \mathbb{C}$ and $k \in \mathbb{R}, k > 0, k \neq 1$ is a circle. When $k = 1$, the locus is a perpendicular bisector.

Circle theorems can also be used to determine more complex loci:

- The locus of points z that satisfy $\arg\left(\frac{z-a}{z-b}\right) = \theta$, with $\theta \in \mathbb{R}, \theta > 0$ and $a, b \in \mathbb{C}$, is an arc of a circle with endpoints at the points of the complex number a, b. Again, these endpoints aren't included in the locus.

Recall from previous complex number work that $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$, so $\arg\left(\frac{z-a}{z-b}\right) = \arg(z - a) - \arg(z - b) = \theta$. From circle theorems, we know that any point that satisfies this equation must lie on a circle, therefore the locus is the arc of a circle drawn anticlockwise from a to b.

- If $\theta < \frac{\pi}{2}$, then the locus is a major arc of the circle (covers over half of the circumference)
- If $\theta > \frac{\pi}{2}$, then the locus is a minor arc of the circle
- If $\theta = \frac{\pi}{2}$, then the locus is a semicircle

The Cartesian equation of the circle can be found both geometrically, using circle theorems and other angle rules, or algebraically.

Example 1: Given that $\arg\left(\frac{z-3}{z-2}\right) = \frac{\pi}{3}$, find, using an algebraic method, the Cartesian equation for the locus of $P(x, y)$ which is represented by z on an Argand diagram.

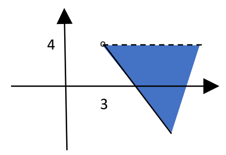
Substitute $z = x + iy$, as we are looking at the equation algebraically, we only need to consider the real and imaginary parts.	$\frac{z-3}{z-2} = \frac{x-3+iy}{x-2+iy}$
Realise the denominator by multiplying the numerator and denominator by $(x - 2 + iy)^*$.	$= \frac{(x-3+iy)(x-2-iy)}{(x-2+iy)(x-2-iy)}$ $= \frac{x^2-2x-iyx-3x+6+3iy+iyx-2iy+y^2}{x^2-2x-iyx-2x+4+2iy+iyx-2iy+y^2}$ $= \frac{x^2-5x+6+y^2+iy}{x^2-4x+4+y^2}$
Complete the square on the denominator and separate the real and imaginary parts.	$= \frac{x^2-5x+6+y^2}{(x-2)^2+y^2} + \left(\frac{y}{(x-2)^2+y^2}\right)i$ So, as given by the question, $\arg\left(\frac{x^2-5x+6+y^2}{(x-2)^2+y^2} + \left(\frac{y}{(x-2)^2+y^2}\right)i\right) = \frac{\pi}{3}$
If $\arg z = \theta$, then $\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)} = \tan\theta$. (Imagine plotting the point z on an Argand diagram- the angle that is made by the half-line from the origin and the real line is the argument and can be calculated using trigonometry).	$\tan\frac{\pi}{3} = \sqrt{3}$ $\frac{y}{(x-2)^2+y^2} = \frac{x^2-5x+6+y^2}{(x-2)^2+y^2}(\sqrt{3})$ $\Rightarrow y = \sqrt{3}(x^2-5x+6+y^2)$
	$\Rightarrow \frac{y}{\sqrt{3}} = x^2-5x+6+y^2$ $\Rightarrow x^2-5x+6+y^2-\frac{y}{\sqrt{3}} = 0$
Complete the square and rearrange into the normal Cartesian form of a circle.	$(x-\frac{5}{2})^2 - \frac{25}{4} + (y-\frac{\sqrt{3}}{6})^2 - \frac{1}{12} + 6 = 0$ $(x-\frac{5}{2})^2 + (y-\frac{\sqrt{3}}{6})^2 = \frac{1}{3}$
We have calculated the equation of the entire circle that the locus is on, but now we must use geometric consideration to work out what part of the circle we are looking for.	$(x-\frac{5}{2})^2 + (y-\frac{\sqrt{3}}{6})^2 = \frac{1}{3}$ Where $y > 0$, since the locus is the major part of the circle that lies above the real axis since $\theta\frac{\pi}{3} < \frac{\pi}{2}$.

Regions in an Argand diagram

Inequalities can be used to define regions in an Argand diagram:

- The inequality $\theta_1 \leq \arg(z - z_1) \leq \theta_2$ represents a region in an Argand diagram that is enclosed by the two half-lines defined by $\arg(z - z_1) = \theta_1$ and $\arg(z - z_1) = \theta_2$. As the inequalities includes the $\leq, \geq$ signs, the half lines are included in the diagram, and represented by a solid line. If the $>, <$ signs are included, then the respective half line is not included and is represented by a dotted line.

Example 2: Sketch the region represented by the inequality $\frac{-\pi}{4} \leq \arg(z - (3 + 4i)) < 0$.

The initial half line is represented by $\arg(z - (3 + 4i)) = \frac{-\pi}{4}$, and is drawn with a solid line. The terminal half-line is represented by $\arg(z - (3 + 4i)) = 0$ and is drawn with a dashed line. Shade in the required region.	

These regions can also be defined using set notation, and using different specifications, such as the modulus (which would give a region of a circle).

Transformations of the Complex Plane

Transformations can take the simple loci that we have explored from one complex plane (the z-plane) to another (the w-plane). The transformation will be defined by a function relating $z = x + iy$ to $w = u + iv$ and will map points from the z-plane to the w-plane. You should be able to recognise the formulae for translations, enlargements, and rotations.

- $w = z + a + ib$ represents a translation by the vector $\binom{a}{b}$, where $a, b \in \mathbb{R}$
- $w = kz$, $k \in \mathbb{R}$, represents an enlargement of scale factor k with centre (0,0)
- $w = e^{i\theta}z$ represents an anticlockwise rotation about the origin of angle θ.

You should also be able to recognise compound transformations, for example the transformation formula $w = iz + 3 - i$ represents an anticlockwise rotation through $\frac{\pi}{2}$ about the origin followed by a translation by the vector $\binom{3}{-1}$.

Example 3: A transformation from the z-plane to the w-plane is given by $w = 3z + 2 + 3i$. Describe the locus of w and give its Cartesian equation when z lies on the circle with Cartesian equation $x^2 + y^2 = 25$.

Recognise that $w = 3z$ represents an enlargement of scale factor 3. The circle now has a radius with value 15.	$u^2 + v^2 = 225$
$w = z + 2 + 3i$ represents a translation by the vector $\binom{2}{3}$.	$(u-2)^2 + (v-3)^2 = 225$

It is important to note that in the z-plane the Cartesian form will use the variables x, y and in the w-plane it will be in terms of u and v.

There is another type of transformation that you should know, called a Möbius transformation, which are of the form, $w = \frac{az+b}{cz+d}$, $a, b, c, d \in \mathbb{C}$.

Example 4: A transformation from the z-plane, where $z = x + iy$, to the w-plane, where $w = u + iv$, is given by $w = \frac{4iz+3i}{z-1}$, $z \neq 1$. Find the image of the z-plane circle $|z| = 1$ in the w-plane.

Rearrange the transformation equation to make z the subject.	$w = \frac{4iz+3i}{z-1} \Rightarrow w(z-1) = 4iz + 3i$ $\Rightarrow wz - w = 4iz + 3i \Rightarrow wz - 4iz = 3i + w$ $\Rightarrow z(w - 4i) = 3i + w \Rightarrow z = \frac{3i+w}{w-4i}$
Take the modulus of each side of the equation.	$\lvert z\rvert = \left\lvert\frac{3i+w}{w+4i}\right\rvert$
Use the expression $\left\lvert\frac{z_1}{z_2}\right\rvert = \frac{\lvert z_1\rvert}{\lvert z_2\rvert}$, and the fact that $\lvert z\rvert = 1$.	$1 = \frac{\lvert 3i+w\rvert}{\lvert w+4i\rvert}$ $\lvert w+4i\rvert = \lvert w+3i\rvert$
Using the previous work on loci, we know that the locus of points $\lvert z - z_1\rvert = \lvert z - z_2\rvert$ is the perpendicular bisector of the line connecting the points z_1 and z_2.	The locus of points satisfying $\lvert w+4i\rvert = \lvert w+3i\rvert$ is the perpendicular bisector of the points connecting $-4i$ and $-3i$, namely the line $v = -3.5$.

Example 5: A transformation T of the z-plane to the w-plane is given by $w = \frac{2iz-4i}{1+z}$, $z \neq -1$. If z lies on the imaginary axis, find the image on the w-plane.

Rearrange the transformation equation to make z the subject of the equation.	$w = \frac{2iz-4i}{1+z} \Rightarrow w(1+z) = 2iz - 4i$ $\Rightarrow w + wz = 2iz - 4i \Rightarrow wz - 2iz = -w - 4i$ $\Rightarrow z(w - 2i) = -w - 4i \Rightarrow z = \frac{-w-4i}{w-2i}$
Rewrite w as $u + iv$.	$z = \frac{-u-iv-4i}{u+iv-2i} \Rightarrow z = \frac{-u-(v+4)i}{u+(v-2)i}$
Realise the denominator (multiply the numerator and denominator by the complex conjugate of the denominator).	$z = \frac{-u-(v+4)i}{u+(v-2)i} \times \frac{u-(v-2)i}{u-(v-2)i}$ $z = \frac{-u^2+(vu-2u)i-(vu+4u)i-(v^2+2v-8)}{u^2-(vu-2u)i+(vu-2u)i+(v^2-4v+4)}$ $z = \frac{-u^2-6ui-v^2-2v+8}{u^2+v^2-4v+4}$
Group together the real and imaginary parts.	$z = \frac{-u^2-(v+4)(v-2)}{u^2+(v-2)^2} - \left(\frac{6u}{u^2+(v-2)^2}\right)i$
Rewrite z as $x + iy$ and equate the real and imaginary parts.	$z = x + iy$ As z lies on the imaginary axis, $x = 0$, $0 + yi = \frac{-u^2-(v+4)(v-2)}{u^2+(v-2)^2} - \left(\frac{6u}{u^2+(v-2)^2}\right)i$ $\Rightarrow 0 = \frac{-u^2-(v+4)(v-2)}{u^2+(v-2)^2}$ $\Rightarrow -u^2-(v+4)(v-2) = 0 \Rightarrow u^2+v^2+2v = 8$ So the image lies on the circle with equation $u^2 + v^2 + 2v = 8$.

Example 6: Deduce the Cartesian equation of the curve $2|z + 3| = |z - 3|$.

Rewrite z as $x + iy$ and group the real and imaginary parts.	$2\lvert x+iy+3\rvert = \lvert x+iy-3\rvert$ $2\lvert(x+3)+iy\rvert = \lvert(x-3)+iy\rvert$
Recall that $\lvert z\rvert = \sqrt{x^2+y^2}$, so we can square both sides and then remove the modulus sign- don't be tempted to square as you would $(x+y)^2$, for example.	$2^2((x+3)^2+y^2) = (x-3)^2+y^2$
Expand the brackets and simplify.	$4(x^2+6x+9+y^2) = x^2-6x+9+y^2$ $\Rightarrow 4x^2+24x+36+4y^2 = x^2-6x+9+y^2$ $\Rightarrow 3x^2+30x+3y^2+27 = 0$ $\Rightarrow x^2+10x+y^2+9 = 0$
Complete the square.	$(x+5)^2 - 25 + y^2 + 9 = 0$
Rearrange into the standard equation of a circle.	$(x+5)^2 + y^2 = 16$ So the locus is a circle of radius 4 centred at $(-5,0)$